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General Characterization Theorem for First-Order Super-Intuitionistic Modal Logics

by

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Abstract

The author establishes here a general characterization result on first-order super-intuitionistic modal logics (including first-order Boolean modal logics) and classes of frames, introducing canonical frames. The result, which deals with propositional axioms, is a first-order version of that on propositional logics in his previous paper under some restrictions.

Introduction

In our [17], extending the results in Goldblatt [6] and Sahlqvist [13] Sect. 5, we established general characterization results on (super-) intuitionistic modal propositional logics and classes of frames. Hence it is natural to consider the following problem:

Do these results hold for first-order super-intuitionistic modal logics?

We present here a solution to the problem. We however regret that the general characterization result in the present paper is not applicable to some of propositional axioms to which the results in [17] are applicable, while we have not found explicit counter-example for which the intended characterization fails.

We restrict ourselves to normal mono-modal logics. But it is just a trivial job to extend our methods and results to multi-modal logics. We may also extend some of them to regular, quasi-normal, or quasi-regular logics, using a variant of Segerberg's technic ([14] Vol. 3); See also Routley [12].

In the present paper exept Appendix, we investigate so-called "constant-domain" frames and logics which contain Barcan's axiom (B) (BF in Hughes & Cresswell [8] Chap. 8 p. 144; essentially due to Barcan [1], as noted there) on modal quantification, and Goernemann's axiom (D) (due to Umezawa [16], to which Goernemann [5] missed referring) on non-modal quantification. In Section 1 we state the minimum basic definitions, postponing as Section 3 the other basic definitions and facts needed in Sections 4 and 5. Section 2 is devoted to the general characterization result of ours (1984), namely the essential contribution of the present paper. In Section 4 we define canonical frames, and in Section 5 we establish the fundamental lemmas used in Section 2. In Appendix we mention without proof our result (1985) on logics without the axioms (D) or (B) and frames with varying domains under the inclusion requirement.

Sections 3, 4, and 5 do not depend on Section 2 at all. Section 2 depends on all other sections. If a reader wants to read in the logical order, he or she has to insert Sections 3, 4, and 5 into, say, the end of Section 1.

We assume readers to be familiar with [17] and with [18] § 1.

§ 1. Preliminaries

For the basic definitions concerning the (first-order modal) language and logics, we refer to [18] § 1. (As in [18] we omit every overscore which distinguishes a diagram constant from its denotation.) We however require in the present paper the following: The ground language must be countable. (Uncountable cases may be managed with compactness, which is out of scope of the present paper.)

We also employ the notations and the definitions in [17] Sect. 1.1, exept a renaming (see (1.3) in the present paper).

(1.1) DEFINITIONS. A logic (first-order super-intuitionistic modal logic; see [18] (1.3)) is *normal* if it satisfies the following two conditions:

(1) If a formula A is contained in it, then so is $\Box A$;

(2) It contains the formula

$$\Box p \wedge \Box(p \rightarrow q) \rightarrow \Box q\,,$$

for every propositional variables p, q.

A logic is a *BD-logic* if it contains the following formulas (B) and (D):

(B) *Barcan's axiom*

$$\forall v \Box p(v) \rightarrow \Box \forall v p(v)\,,$$

for every unary predicate variable p and individual bound variable v;

(D) *Goernemann's axiom* (in fact due to Umezawa)

$$\forall v[q \vee p(v)] \rightarrow q \vee \forall v p(v)\,,$$

for every propositional variable q, unary predicate variable p, and individual bound variable v.

Remark. Every logic (in our definition) contains the converses of the formulas (B) and (D).

(1.2) DEFINITIONS (supplements to [18] § 1). Let U be a non-empty set. A *U-sentence* is a sentence in the diagram language for U.

A formula A in the diagram language for U is a *v/a-sentencializable U-formula* if A is v/a-quantifiable ([18] (1.2)) and contains no occurrence of any individual free variables other than a.

(1.3) *Remark.* We rename a "frame" defined in [17] (1.1.5) a "frame without domain" in the present paper.

(1.4) DEFINITION. Let U be a non-empty set and $(X, R_0, R_{0,1,0})$ a frame without domain. We call the pair $((X, R_0, R_{0,1,0}), U)$ a *frame.*

(1.5) DEFINITION. Let $((X, R_0, R_{0,1,0}), U)$ be a frame and $(\boldsymbol{O}(X, R_0), L)$ be the normal mHa constructed from $(X, R_0, R_{0,1,0})$ as [17] (4.2.2) with (4.1.5). A *valuation* $[\![-]\!]$ *on* the frame $((X, R_0, R_{0,1,0}), U)$ assigns

$$[\![p(u_1, \cdots, u_n)]\!] \in \boldsymbol{O}(X, R_0)$$

for each $n \geqslant 0$, each n-ary predicate variable p, and each $u_1, \cdots, u_n \in U$; And we inductively define

$$[\![A]\!] \in \boldsymbol{O}(X, R_0)$$

for every U-sentence A as follows.

Falsum: $[\![\bot]\!] = \varnothing$.

Propositional connectives:

$$[\![A \rightharpoonup B]\!] = [\![A]\!] \rightharpoonup [\![B]\!]\ ; \quad \text{(distinguish two "}\rightharpoonup\text{"s!)}$$

$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]\ ;$$

$$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]\ ;$$

$$[\![\Box A]\!] = L[\![A]\!]\ .$$

Quantifiers:

$$[\![\forall v A(v/a)]\!] = \bigcap \{[\![A(u/a)]\!] \mid u \in U\}\ ;$$

$$[\![\exists v A(v/a)]\!] = \bigcup \{[\![A(u/a)]\!] \mid u \in U\}\ ;$$

where A must be a v/a-sentencializable U-formula.

(1.6) DEFINITIONS. Let $((X, R_0, R_{0,1,0}), U)$ be a frame and A a U-sentence. A valuation $[\![-]\!]$ on the frame $((X, R_0, R_{0,1,0}), U)$ *validates* the sentence A if $[\![A]\!] = X$.

The frame $((X, R_0, R_{0,1,0}), U)$ *validates* the sentence A if every valuation on it validates the sentence A.

(1.7) PROPOSITION. *Let $((X, R_0, R_{0,1,0}), U)$ be a frame. The set of all formulas in the ground language whose $\forall$-closures are validated by $((X, R_0, R_{0,1,0}), U)$ is a normal BD-logic.*

Proof. Straightforward.

Remark. We call it the *logic determined by* the frame.

(1.8) PROPOSITION. *The intersection of any family of normal BD-logics is a normal BD-logic.*

Proof. Straightforward.

(1.9) DEFINITION (cf. [17] (1.1.7)). A modal Heyting polynomial in variables $v_1, \cdots, v_n$ is *purely affirmative* if it is constructed (at most) from the constants (see Remark 2 below) and the variables $v_1, \cdots, v_n$ with operators "$\cap$", "$\cup$", "L".

Remark 1. In the definition of affirmative polynomials ([17] (1.1.7)), we permitted to apply "$\neg L^p \neg$" ($p \geqslant 0$).

Remark 2. In the definition above, "the constants" mean any modal Heyting polynomials containing no variables (e.g. "0", "L 0"). The atomic constant is "0" only. ("1" should be considered as an abbreviation for "$\neg 0$", i.e. "$0 \rightharpoonup 0$".)

§ 2. General characterization theorem

In the present section we establish first-order versions of [17] (3.1.1), (3.2.2) under some restrictions.

(2.1) DEFINITION (cf. [17] (1.1.6)). A condition (C^*) on a frame *characterizes* a condition (C) on a normal BD-logic if the following (a) and (b) hold:

(a) Every normal BD-logic determined (cf. (1.7)) by a frame satisfying the condition (C^*) satisfies the condition (C);

(b) Every canonical frame (see § 4 for details) for a normal BD-logic satifying the condition (C) satisfies the condition (C^*).

Remark. Generally speaking, every normal *BD*-logic includes (set-theoretically) the logic determined by its canonical frame (see (4.6)).

(2.2) PROPOSITION. *Let* $n \geqslant 0$; $A_1, \cdots, A_n$ *be formulas in the ground language,* (C_i) *be the condition, on a normal BD-logic, such that the logic contains the formula* A_i, *for* $1 \leqslant i \leqslant n$; *and suppose the condition* (C_i) *is characterized by a condition* (C_i^*) *on a frame, for* $1 \leqslant i \leqslant n$. *Then the normal BD-logic generated* (*cf.* (1.8)) *by* $\{A_i \mid 1 \leqslant i \leqslant n\}$ *is identical to the logic determined by its canonical frame, and sound and sound and complete* (*defined as usual*) *with respect to the class of frames satisfying all* (C_i^*) *for* $1 \leqslant i \leqslant n$.

Proof. Straightforward.

(2.3) DEFINITION. We define as usual the translation of a modal Heyting polynomial (see (1.9) Remark 2 for "1") to a propositional sentence. We write a translated sentence $|\alpha|(p_1, \cdots, p_n)$ for a polynomial $\alpha(v_1, \cdots, v_n)$, where the variables (in both) must be fully indicated and distinct one another.

(2.4) GENERAL CHARACTERIZATION THEOREM (cf. [17] (3.1.1)). *Let* $n, k \geqslant 0$;

$m(j)\geqslant 0$ for $1\leqslant j\leqslant k$;

$r(j)\geqslant 0$ and $\alpha_j(\mathfrak{v}_1, \cdots, \mathfrak{v}_n)$ be purely affirmative polynomials (in variables $\mathfrak{v}_1, \cdots, \mathfrak{v}_n$) for $0\leqslant j\leqslant k$;

$l(j, d)=0$ or 1 and $1\leqslant s(j, d)\leqslant n$ for $0\leqslant j\leqslant k$ and $1\leqslant d\leqslant r(j)$.
Then the following condition (1) on a frame $((X, R_0, R_{0,1,0}), U)$ characterizes the condition (2) below on a normal BD-logic.

(1) $\forall y_0\in X(\forall y_j\in R_{0,1,0}{}^{m(j)}(y_0))_{1\leqslant j\leqslant k}$

$$\left[\bigvee_{j=0}^{k} y_j\in\alpha_j(N_1(y_0, \cdots, y_k), \cdots, N_n(y_0, \cdots, y_k))\right],$$

where

$$N_i(y_0, \cdots, z_k)=\bigcup_{j=0}^{k}\ \bigcup_{\substack{d=1,\\ s(j,d)=i}}^{r(j)} R_{0,1,0}{}^{l(j,d)}(y_j)$$

for $1\leqslant i\leqslant n$.

(2) *The logic contains (the $\forall$-closures of the substitution-instances of) the following:*

$$F_0(p_1, \cdots, p_n)\rightarrow(|\alpha_0|(p_1, \cdots, p_n)\vee\bigvee_{j=1}^{k}\ \Box^{m(j)}(F_j(p_1, \cdots, p_n)\rightarrow|\alpha_j|(p_1, \cdots, p_n))),$$

where

$$F_j(p_1, \cdots, p_n)\quad \text{is}\quad \left(\bigwedge_{d=1}^{r(j)}\Box^{l(j,d)}p_{s(j,d)}\right)\quad \text{for}\quad 0\leqslant j\leqslant k.$$

Remark 1. Recall our convention ([17] (1.1.5)):

$$R_{0,1,0}{}^{0}=R_0\,.$$

Proof. Similar to [17] (3.1.1). Apply the lemmas in § 5 in the present paper instead of those in [17] Chap. 5.

Remark 2. In the following examples we omit to repeat the leading clause.

(2.5) *Example* (cf. [17] (3.1.3)). Let $n\geqslant 0$; $m(i)\geqslant 0$, $l(i)=0$ or 1 for $1\leqslant i\leqslant n$; and $\alpha(\mathfrak{v}_1, \cdots, \mathfrak{v}_n)$ be a purely affirmative polynomial in variables $\mathfrak{v}_1, \cdots, \mathfrak{v}_n$.

(1) $\forall x\in X(\forall y_i\in R_{0,1,0}{}^{m(i)}(x))_{1\leqslant i\leqslant n}$

$$[x\in\alpha(R_{0,1,0}{}^{l(1)}(y_1), \cdots, R_{0,1,0}{}^{l(n)}(y_n))]\,.$$

(2) The logic contains the the following:

$$\left(\bigvee_{i=1}^{n} \Box^{m(i)} \neg \Box^{l(i)} p_i\right) \vee |\alpha|(p_1, \cdots, p_n).$$

(2.6) *Example* (cf. [17] (3.1.4)). Let $n \geqslant 0$; $l(i)=0$ or 1 for $1 \leqslant i \leqslant n$; and $\alpha(v_1, \cdots, v_n)$ be a purely affirmative polynomial in variables $v_1, \cdots, v_n$.

(1) $\forall x \in X\, [x \in \alpha(R_{0,1,0}{}^{l(1)}(x), \cdots, R_{0,1,0}{}^{l(n)}(x))]$.

(2) The logic contains the following:

$$\left(\bigwedge_{i=1}^{n} \Box^{l(i)} p_i\right) \rightarrow |\alpha|(p_1, \cdots, p_n).$$

(2.7) *Example*. Let $k \geqslant 0$; $l=0$ or 1.

(1) $\forall x \in X\, [R_{0,1,0}{}^{k}(x) \subseteq R_{0,1,0}{}^{l}(x)]$.

(2) The logic contains the following:

$$\Box^l p \rightarrow \Box^k p.$$

(2.8) *Examples*.

(1) The analogues of [17] (3.1.6), (3.1.7), (3.1.8), (3.1.9), (3.1.13) hold in the present situation.

(2) The analogues of [17] (3.1.10), (3.1.11), (3.1.12) hold in the present situation, under the following restriction: $l=0$ or 1.

(2.9) *Problem*. How about [17] (3.1.5), (3.1.14) and the results in [17] Chap. 2? Prove or find counter-examples, in the present situation.

Remark. We state another open problem in (5.3).

(2.10) DEFINITION. We define *purely cute polynomials*, by modifying the definition of cute polynomials ([17] (3.2.1)) as follows:

(a) In (1), replace "affirmative" by "purely affirmative" (see (1.9));

(b) In (2), let $q=0$, and $l(d)=0$ or 1 (for each d);

(c) And throughout (1), (2), (3), replace "cute" by "purely cute".

(2.11) THEOREM (cf. [17] (3.2.2), [13] Theorems 8 and 19). *Let $n \geqslant 0$; and $\alpha(v_1, \cdots, v_n)$ be a purely cute polynomial in variables $v_1, \cdots, v_n$. The following condition* (**) *on a normal BD-logic is characterized by a condition* (*) *on a frame, where* (*) *is effectively obtainable from* (**):

(**) *The logic contains the sentence*

$$|\alpha|(p_1, \cdots, p_n).$$

Proof. Similar to [17] (3.2.2). Apply the lemmas in §5 of the present paper instead of those in [17] Chap. 5.

(2.12) *Examples*. The analogues of [17] (3.2.3), (3.2.4) hold in the present situation.

§ 3. Preliminaries (continuation)

We prepare here basic definitions and lemmas (proofs omitted), for canonical construction. Throughout the present section, fix a logic arbitrarily; "a language" means "the ground language or the diagram language for some non-empty set".

The present section does not depend on § 2 at all.

(3.1) DEFINITION (Segerberg [14] pp. 9–10). Let T be a set of sentences in a language, and A be a formula in the language. We write

$$T \vdash A \quad \text{(in the logic)}$$

if there exist some finite subset S of T such that $\bigwedge S \rightharpoonup A$ is provable (see [18] (1.4)) in the logic.

Remark. For every sentence A, B,

$$\{A, B\} \vdash A \wedge B\,,$$

$$\{A, A \rightharpoonup B\} \vdash B\,.$$

(3.2) LEMMA (Segerberg [14] p. 10). *Let T be a set of sentences in a language, and A, B be sentences in the language. Then*:

$$(T \cup \{A\} \vdash B) \qquad \textit{if and only if} \quad (T \vdash A \rightharpoonup B)\,.$$

Remark. We do not call the lemma "Deduction theorem", because (1) we defined "$\vdash$" not by so-called "deduction", and (2) contrary to the present lemma, so-called "Deduction theorem" fails for usually considered modal logics.

(3.3) LEMMA. *Let T be a set of sentences in a language, and A, B be sentences in the language. If $(T \nvdash A)$, then $(T \nvdash A \vee B)$ or $(T \cup \{B\} \nvdash A)$.*

(3.4) LEMMA. *We assume here that the logic is normal. Let T be a set of sentences in a language, and B be a sentence in the language.*

$$\textit{If} \quad (T \vdash B), \qquad \textit{then} \quad (\{\Box A \mid A \in T\} \vdash \Box B)\,.$$

(3.5) DEFINITIONS. Let T be a set of sentences in a language. The set T is a *theory* (*over* the logic) *in* the language if the set T contains every sentence A, in the language, such that $T \vdash A$ in the logic.

The set T is *prime* (*over* the logic) *in* the language if it satisfies the following two conditions:

(1) $T \nvdash \bot$;

(2) For every sentences A, B in the language,

$$\text{if} \quad (T \vdash A \vee B)\,, \qquad \text{then} \quad (T \vdash A) \quad \text{or} \quad (T \vdash B)\,.$$

(3.6) DEFINITIONS. Let U be a non-empty set, and T a set of U-sentences (see

(1.2)). The set T is a *co-semi-Henkin set on U* (*over* the logic) if it satisfies the following condition (1):

(1) For every v/a-sentencializable U-formula A,

$$\text{if} \quad (T \vdash A(u/a)) \qquad \text{for all} \quad u \in U,$$

$$\text{then} \quad (T \vdash \forall v A(v/a)).$$

The set T is a *Henkin set on U* (*over* the logic) if it is a co-semi-Henkin set on U and satisfies the following condition (2):

(2) For every v/a-sentencializable U-sentence A,

$$\text{if} \quad (T \vdash \exists v A(v/a)),$$

$$\text{then} \quad (T \vdash A(u/a)) \qquad \text{for some} \quad u \in U.$$

The set T is a *prime Henkin theory on U* (*over* the logic) if the set T is a Henkin set on U and prime theory in the diagram language for U.

Remark 1. The condition (2) above defines *semi-Henkin sets.* The notion of *prime semi-Henkin theories* is useful in studying logics without Barcan's or Goernemann's axiom.

Remark 2. The origin of the notion of co-semi-Henkin sets is omega-completeness in arithmetic (cf. Henkin [7] p. 1).

Remark 3. The empty set is a co-semi-Henkin set on any (non-empty) set.

(3.7) LEMMA. *Let U be a non-empty set, and T be a co-semi-Henkin set on U. For every U-sentence B and for every v/a-sentencializable U-formula A,*

$$\textit{if} \quad (T \cup \{A(u/a)\} \vdash B) \qquad \textit{for all} \quad u \in U,$$

$$\textit{then} \quad (T \cup \{\exists v A(v/a)\} \vdash B).$$

(3.8) LEMMA. *Let U be a non-empty set and B be a U-sentence. If T is a co-semi-Henkin set on U, then so is $T \cup \{B\}$.*

(3.9) LEMMA. *We assume here that the logic contains Goernemann's axiom. Let U be a non-empty set, T be a co-semi-Henkin set on U, B be a U-sentence, and A be a v/a-sentencializable U-formula.*

$$\textit{If} \quad (T \vdash B \vee A(u/a)) \qquad \textit{for all} \quad u \in U,$$

$$\textit{then} \quad (T \vdash B \vee \forall v A(v/a)).$$

(3.10) LEMMA. *We assume here that the logic is normal and contains Barcan's axiom. Let U be a non-empty set. If T be a co-semi-Henkin set on U, then so is the set $\{A \mid T \vdash \Box A\}$.*

§ 4. Canonical frames

In the present section we construct canonical frames for normal BD-logics. The technic employed here was introduced by R. H. Thomason in [15] pp. 57–62 for Boolean modal logics containing Barcan's axiom, used by McArthur & Leblanc in [9] for Boolean tense logics containing Barcan's axiom bidirectionally, and adopted by Ono in [11] Sect. 3 for super-intuitionistic (non-modal) logics containing Goernemann's axiom. We feel that this technic is more elegant than that introduced by Hughes & Cresswell in [8] Chap. 9 and used by Freeman in [3] pp. 537–539.

The present section does not depend on § 2 at all.

Throughout the present section, fix a normal BD-logic arbitrarily; U is a countably infinite set; X is the set of all prime Henkin theories on U over the logic;

$$[\![A]\!]_0 = \{x \in X \mid A \in x\}$$

for every U-sentence A ($[\![—]\!]_0$ will be proved to be a valuation);

$$\begin{aligned} R_0 &: X \longrightarrow \boldsymbol{P}(X) \\ \text{s.t.}\quad & x \longmapsto \{y \in X \mid x \subseteq y\} = \bigcap \{[\![A]\!]_0 \mid A \in x\}\,, \\ R_{0,1,0} &: X \longrightarrow \boldsymbol{P}(X) \\ \text{s.t.}\quad & x \longmapsto \{y \in X \mid \{A \mid (\Box A) \in x\} \subseteq y\} = \bigcap \{[\![A]\!]_0 \mid (\Box A) \in x\}\,. \end{aligned}$$

(4.1) PROPOSITION. *The pair* $((X, R_0, R_{0,1,0}), U)$ *is a frame.*

Proof. Straightforward.

Remark. We call the pair a *canonical frame* (*for* the normal BD-logic). It is unique for the logic upto the (unessential) choice of the countably infinite set U.

(4.2) THEOREM. *Let* T *be a co-semi-Henkin set on* U, B *be a* U*-sentence. If* $T \nvdash B$, *then there exists* $x \in X$ *such that* $T \subseteq x$ *and* $B \notin x$.

Remark 1. This theorem (and its consequences) depends on Goernemann's axiom (cf. (3.9)) as well as on the cardinality (countably infinite) of the diagram language for U.

Remark 2. The following proof is but a revision for Ono [11] pp. 51–52.

Proof. Fix an enumeration of all U-sentences. We first construct inductively two sequences

$$F_0, F_1, \cdots,$$

$$J_0, J_1, \cdots$$

of finite sets of U-sentences, requiring

$$T \cup F_n \nvdash \bigvee J_n$$

for each n. By virtue of (3.8), the set $T \cup F_n$ will be a co-semi-Henkin set on U, for each n.

Basis. $F_0 = \varnothing$, $J_0 = \{B\}$.

Induction step.

Case 1: The outermost symbol of the n-th sentence A is neither $\forall$ nor $\exists$. From the induction hypothesis and (3.3), it follows that

$$T \cup F_n \nVdash \bigvee (J_n \cup \{A\}) \qquad \text{or} \qquad T \cup F_n \cup \{A\} \nVdash \bigvee J_n .$$

On former, we let

$$F_{n+1} = F_n , \qquad J_{n+1} = J_n \cup \{A\} ;$$

Otherwise

$$F_{n+1} = F_n \cup \{A\} , \qquad J_{n+1} = J_n .$$

Case 2: The n-th sentence is of the form $\forall v A(v/a)$, where A is v/a-sentencializable.

Subcase 2.1: We have $u \in U$ such that

$$T \cup F_n \nVdash \bigvee (J_n \cup \{A(u/a)\}) .$$

This implies that

$$T \cup F_n \nVdash \bigvee (J_n \cup \{A(u/a), \forall v A(v/a)\}) .$$

We thus let

$$F_{n+1} = F_n , \; J_{n+1} = J_n \cup \{A(u/a), \forall v A(v/a)\} .$$

Subcase 2.2: For all $u \in U$,

$$T \cup F_n \vdash \bigvee (J_n \cup \{A(u/a)\}) .$$

By virtue of (3.9),

$$T \cup F_n \vdash \bigvee (J_n \cup \{\forall v A(v/a)\}) .$$

From the induction hypothesis and (3.3), it follows that

$$T \cup F_n \cup \{\forall v A(v/a)\} \nVdash \bigvee J_n .$$

We thus let

$$F_{n+1} = F_n \cup \{\forall v A(v/a)\}, \; J_{n+1} = J_n .$$

Case 3: The n-th sentence is of the form $\exists v A(v/a)$, where A is v/a-sentencializable.

Subcase 3.1: We have $u \in U$ such that

$$T \cup F_n \cup \{A(u/a)\} \nVdash \bigvee J_n .$$

This implies that

$$T \cup F_n \cup \{A(u/a), \exists v A(v/a)\} \nvdash \bigvee J_n .$$

We thus let

$$F_{n+1} = F_n \cup \{A(u/a), \exists v A(v/a)\} , \qquad J_{n+1} = J_n .$$

Subcase 3.2: For all $u \in U$,

$$T \cup F_n \cup \{A(u/a)\} \vdash \bigvee J_n .$$

By virtue of (3.7),

$$T \cup F_n \cup \{\exists v A(v/a)\} \vdash \bigvee J_n .$$

From the induction hypothesis and (3.3), it follows that

$$T \cup F_n \nvdash \bigvee (J_n \cup \{\exists v A(v/a)\}) .$$

We thus let

$$F_{n+1} = F_n , \qquad J_{n+1} = J_n \cup \{\exists v A(v/a)\} .$$

To complete the proof of the lemma, we let

$$x = \bigcup_{n \geqslant 0} F_n ,$$

which has the desired properties. //

(4.3) LEMMA (cf. [15] p. 59 L3.). *Let B be a U-sentence, and $x \in X$. Then*:

$$R_{0,1,0}(x) \subseteq [\![B]\!]_0 \quad \text{if and only if} \quad x \in [\![B]\!]_0 .$$

Remark. This lemma depends on normality and Barcan's axiom (cf. (3.10)).

Proof. Apply (3.4), (3.10), and (4.2). //

(4.4) LEMMA. *Let A, B be U-sentences, and $x \in X$. Then*:

$$x \in [\![A \rightharpoonup B]\!]_0 \quad \text{if and only if} \quad R_0(x) \cap [\![A]\!]_0 \subseteq [\![B]\!]_0 .$$

Proof. Apply (3.2), (3.8), and (4.2).

(4.5) THEOREM. $[\![-]\!]_0$ *is a valuation* (*see* (1.5)) *on* $((X, R_0, R_{0,1,0}), U)$.

Proof. Immediate from the definition of prime Henkin theories and (4.3), (4.4).

Remark 1. We call $[\![-]\!]_0$ the *canonical valuation* (*for* the logic) *of* U-sentences.

Remark 2 (cf. [18] §2). Let $H' = \{[\![A]\!]_0 \mid A\text{: } U\text{-sentence}\}$.

The pair (H', L') is a sub-mHa of $(\boldsymbol{O}(X, R_0), L)$, where the operator L' is the restriction of L. We can construct a first-order enhanced mHa $(U, H', L', (Q_n)_{n \geqslant 0})$ as

[18] (2.5) using H' in place of the Lindenbaum algebra. Although we obtained a stronger tool along the line above, we persist in frames in the present paper.

(4.6) PROPOSITION. *Let A be a U-sentence. Then*:

$$\varnothing \vdash A \qquad \text{if and only if} \quad [\![A]\!]_0 = X\,.$$

Proof. Since $\varnothing$ is a co-semi-Henkin set, this follows from (4.2). //

(4.7) THEOREM (not needed later). *Let T, R be sets of sentences in the ground language. If $T \nvdash \bigvee J$ for every finite subset J of R, then there exists $x \in X$ such that $T \subseteq x$ and $x \cap R = \varnothing$.*

Proof. Omitted.

Remark. This theorem concerns strong completeness.

§5. Fundamental lemmas

In the present section we establish fundamental lemmas, used in §2, on canonical frames for normal BD-logics. It must be noted that we do not have any useful lemma in order to obtain a prime Henkin theory (on a set U) included in a non-trivially given set (of U-sentences), nor that in order to obtain a prime Henkin theory (on U) which includes two given sets (of U-sentences).

The present section does not depend on §2 at all. We assume readers to be familiar with [17] Chap. 5, where the fundamental lemmas on Stone representation frames of normal mHa's were established.

Throughout the present section, fix a normal BD-logic arbitrarily; $((X, R_0, R_{0,1,0}), U)$ is a canonical frame for the logic; and

$$H' = \{[\![A]\!]_0 \mid A\colon U\text{-sentence}\}\,,$$

where $[\![\text{—}]\!]_0$ is the canonical valuation (for the logic) of U-sentences.

(5.1) DEFINITIONS. We define nuclear sets and functions as [17] (5.1.1) and (5.2.1) respectively, using H' defined above.

(5.2) LEMMA. *For every $x \in X$, the sets $R_0(x)$ and $R_{0,1,0}(x)$ are nuclear.*

Proof. Immediate.

Remark. This is an extremely restricted version of Basic nuclearity lemma (cf. [17] (5.1.7)).

(5.3) *Basic Nuclearity Problem.* How about $R_{0,1,0}{}^l(x)$ for $l > 1$? Prove nuclearity or find counter-example, in the present situation.

Remark. Affirmative solution (if possible) to this problem will improve the results in §2 of the present paper: We may replace "$=0$ or 1" by "$\geqslant 0$".

(5.4) LEMMA. *Disjunction nuclearity lemma* ([17] (5.2.2)) *and its corollaries* ([17] (5.2.3), (5.2.4)) *hold in the present situation.*

Proof. Similar to those in [17] Chap. 5.

(5.5) LEMMA. *In the present situation, Affirmative nuclearity theorem* ([17] (5.2.9)) *hold for purely affirmative cases.*

Remark. The cause of this restriction is lack of Pseudo-diamond nuclearity lemma.

Proof. Similar to that in [17] Chap. 5.

Appendix

For first-order super-intuitionistic normal modal logics without axiom (D) or (B), and frames with varying domains under the inclusion requirement (see, e.g., [8] Chap. 10, [10] Sect. 3), what we have established (not included in the present paper) is an analogue of [17] (3.1.1) under the following restrictions:

$$k=0\,;\qquad l(0,d),\ \ r(0,d)=0 \text{ or } 1\qquad \text{for each}\ \ d\,;$$

$$\alpha_0\ \ \text{and}\ \ \beta_{0,d}\ (\text{for each } d)\ \text{do not contain}\ \ \text{“}\neg L^p\neg\text{”}\qquad \text{for}\ \ p>1\,.$$

A canonical frame with varying domain contains prime semi-Henkin theories on various sets simultaneously; In particular, $R_0(x)$ may contain conservative extensions of x, even for Boolean modal logics; For every point x, the notion of nuclearity must be relativized to the diagram language for the set of the individuals available at x. Moreover we do not have any useful lemma in order to obtain a prime semi-Henkin theory included in a non-trivially given set of sentences.

Remark 1. In [2] Sect. 4 Bowen claimed completeness of some Boolean modal logics without Barcan's axiom. We however found serious gaps in his proof of Lemma 4.11 (A3, A7, A8, A9, A13, A14), as well as his illegal applications of R3 just before the lemma.

Remark 2. In [4] Theorem 38 Gabbay claimed completeness of some (non-modal) super-intuitionistic logics without Goernemann's axiom. His claim (b) in the theorem depends on [4] Lemma 39. We however found a delicate gap in his proof of the lemma: At [4] p. 405 lines 8–9 Gabbay presupposed, without any written guarantee, that all the constants common in Δ_1 and Δ_2 must be contained in Δ. (Prof. H. Ono has kindly informed me that his statement in [11] p. 49 lines 29–30 on Gabbay's work is incorrect.)

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Added in proof. The author regrets that he had failed to correct at least twenty mistypes and misprints in [17], and one in [18], five in [22]; most harmful in those are the following.

[17] p. 188. In (3.1.14) Remark: For "0" read "1".

[17] p. 193. In (5.1.1) Definitions: For "conditions", read "condition".

[17] p. 193. In (5.1.1) Remark 2: For "work", read "word".

[17] p. 194. In (5.1.6) Proof: For "a", "an", read "the".

[22] p. 60. In (1.3) Definition: For "(Z, R_0)", read "(X, R_0)".

[22] p. 60. In (1.3) Remark 2: Insert the following.
"we define

$$R_0^{-1}: X \longrightarrow \boldsymbol{P}(X) \qquad \text{by} \quad x \longrightarrow \{y \in X \mid x \in R_0(y)\}.$$

Hence."

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